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On the Degree of Weak Convergence of a Sequence of Finite Measures to the Unit Measure under Convexity

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This is a study of the degree of weak convergence under convexity of a sequence of finite measures $\{\mu_j\}_{j\in\mathbb{N}}$ on $\mathbb{R}^k$, $k\geq 1$, to the unit measure δ_{x_0}. Let Q denote a convex and compact subset of $\mathbb{R}^k$, let $f\in C^m(Q)$, $m\geq 0$, satisfy a convexity condition and let μ be a finite measure on Q. Using standard moment methods, upper bounds and best upper bounds are obtained for $|\int_Q f\,d\mu - f(x_0)|$. They sometimes lead to sharp inequalities which are attained for particular μ and f. These estimates are better than the corresponding ones found in the literature.

1. INTRODUCTION

The flavor of this paper is conveyd by Proposition 1. It claims the equivalence of the weak convergence of a sequence of finite measures $\{\mu_j\}_{j\in\mathbb{N}}$ on $[a,b]\subset\mathbb{R}$ to the unit (Dirac) measure δ_{x_0}, where $x_0\in(a,b)$, with the convergence of $\int f\,d\mu_j$ to $f(x_0)$, where $f\in C^m([a,b])$ for some $m\geq 0$ is such that $|f^{(m)}(t)-f^{(m)}(x_0)|$ is convex in t. For this restricted class of functions f we prove quantitative estimates on the above weak convergence.

The main results are Theorems 3, 7 and the multidimensional Theorem 17.

The inequalities established are usually the best possible and are stronger than the corresponding ones obtained from Shisha and Mond [17], Mond and Vasudevan [15], Gonska [7], Anastassiou [1] and others.

Our work is related to the convergence of linear positive operators since, by Riesz's representation theorem, the pointwise convergence of a sequence of linear positive operators $\{L_j\}_{j\in\mathbb{N}}$ to the unit operator I acting on $C([a,b])$, is equivalent to the weak convergence of a sequence of finite measures $\{\mu_j\}_{j\in\mathbb{N}}$ to the unit (Dirac) measure at the given point.

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2. Preliminaries

We start with

Proposition 1. *Let m be an integer $\geqslant 1$. Let $\{\mu_j\}_{j\in\mathbb{N}}$ be a sequence of measures on $[a, b] \subset \mathbb{R}$ with corresponding masses m_j: $0 < m_j \leqslant \tau$ and δ_{x_0} the unit (Dirac) measure at $x_0 \in (a, b)$. Then the following are equivalent:*

(i) $\mu_j \rightrightarrows \delta_{x_0}$ *(weakly)*;

(ii) $\int f\, d\mu_j \to f(x_0)$ *for all* $f \in C^m([a, b])$

such that $|f^{(m)}(t) - f^{(m)}(x_0)|$ *is convex in* t.

Proof. (i)$\Rightarrow$(ii) Obvious [6, p. 316]. In fact (i) implies $\int f\, d\mu_j \to f(x_0)$ for all $f \in C([a, b])$.

(ii)$\Rightarrow$(i) The set of functions $\{1, (t-x_0), (t-x_0)^2\}$ is a subset of $C^m([a, b])$ and for each of them $|f^{(m)}(t) - f^{(m)}(x_0)|$ is a convex function of t.

Therefore, by assumption, for the positive linear functionals $L_j(f) = \int f\, d\mu_j$ we have $L_j(f) \to f(x_0)$ for any $f \in \{1, (t-x_0), (t-x_0)^2\}$.

Since this triplet of functions is a Chebyshev system, by Korovkin's theorem for positive linear functionals [12], we get $\int f\, d\mu_j \to f(x_0)$ for all $f \in C([a, b])$. This implies $\mu_j \rightrightarrows \delta_{x_0}$ (weakly); see [6]. ∎

The following result plays an important role in the proofs of this paper.

Lemma 2. *Let $(V, \|\cdot\|)$ be a real normed vector space and U a star-shaped subset of V with respect to $x_0 \in U$. Let w, h be positive numbers such that $h \leqslant \|t - x_0\|$ for each extreme point $t \not\equiv x_0$ of U. Consider a convex $f: U \to \mathbb{R}$ such that $f(x_0) = 0$ and*

$$|f(x) - f(y)| \leqslant w \qquad \text{if} \qquad \|x - y\| \leqslant h,\ x, y \in U. \tag{2.1}$$

Then the maximal function satisfying the above conditions is

$$\phi(t) = \frac{w}{h}\, \|t - x_0\|, \qquad t \in U,$$

so that

$$f(t) \leqslant \phi(t) \qquad \text{for all} \quad t \in U.$$

Note. If U is convex, then in the lemma we require that the ball $B(x_0, h) \subset U$.

Proof. The function ϕ fulfills all the assumptions of the lemma. Namely, $\phi(x_0) = 0$ and for $x, y \in U$ with $\|x - y\| \leqslant h$, we have $|\phi(x) - \phi(y)| \leqslant w$. Also, one easily sees that ϕ is convex. Next, for $t \in U - \{x_0\}$ satisfying $\|t - x_0\| \leqslant h$, consider $x \in U$ such that

$$t = \left(\frac{h - \|t - x_0\|}{h}\right) x_0 + \frac{\|t - x_0\|}{h} x.$$

Then $\|x - x_0\| = h$. Since f is convex and $f(x_0) = 0$, we have

$$f(t) = f\left(\left(\frac{h - \|t - x_0\|}{h}\right) x_0 + \frac{\|t - x_0\|}{h} x\right) \leqslant \frac{\|t - x_0\|}{h} f(x)$$

and therefore $f(t) \leqslant (\|t - x_0\|/h) f(x)$. Thus

$$f(t) \leqslant \frac{\|t - x_0\|}{h} |f(x) - f(x_0)| \leqslant \frac{\|t - x_0\|}{h} w$$

so

$$f(t) \leqslant \frac{\|t - x_0\|}{h} w \qquad \text{when} \quad \|t - x_0\| \leqslant h. \tag{2.2}$$

Now for $t \in U$ such that $\|t - x_0\| > h$, there is a finite sequence of points $x_1, ..., x_n$ on the line segment tx_0 such that all of $\|x_0 - x_1\|$, $\|x_1 - x_2\|$, $\|x_2 - x_3\|, ..., \|x_n - t\|$ are $\leqslant h$ and $\|x_0 - x_1\| + \|x_1 - x_2\| + \cdots + \|x_n - t\| = \|t - x_0\|$. Furthermore, the function $F(t) = f(t) - f(x_1)$ is convex, $F(x_1) = 0$ and fulfills (2.1). Since $\|t - x_1\| \leqslant h$, by (2.2) we get $f(t) - f(x_1) \leqslant (\|t - x_1\|/h) w$; similarly

$$f(x_1) - f(x_2) \leqslant \frac{\|x_1 - x_2\|}{h} w, ..., f(x_n) - f(x_0) \leqslant \frac{\|x_n - x_0\|}{h} w.$$

Adding up all these inequalities, we find $f(t) \leqslant (\|t - x_0\|/h) w$ when $\|t - x_0\| > h$. The proof is now complete. ∎

3. One Dimensional Results

Theorem 3. *Let $r > 0$, μ a finite measure of mass m on an interval $[a, b]$, $x_0 \in (a, b)$. Set $c(x_0) = max(x_0 - a, b - x_0)$ and*

$$\left(\int |t - x_0|^r \mu(dt)\right)^{1/r} = d_r(x_0), \tag{3.1}$$

and assume $d_r(x_0) > 0$. In order that μ exist, we also assume that

$d_r^r(x_0) \leqslant m \cdot (c(x_0))^r$. Next consider $f\colon [a, b] \to \mathbb{R}$ for which $|f(t) - f(x_0)|$ is convex in t and

$$|f(s) - f(t)| \leqslant w \text{ when } s, t \in [a, b]; \qquad |s - t| \leqslant h. \tag{3.2}$$

Here $0 < h \leqslant \min(x_0 - a, b - x_0)$ and $w > 0$ are fixed.

A best upper bound is given by

$$\left| \left| \int f\, d\mu - f(x_0) \right| - |m - 1| \cdot |f(x_0)| \right| \leqslant \begin{cases} w\, m^{1-(1/r)} \left(\dfrac{d_r(x_0)}{h} \right), & r \geqslant 1, \\ w(c(x_0))^{1-r} \dfrac{d_r^r(x_0)}{h}, & r \leqslant 1. \end{cases} \tag{3.3}$$

Remark 4. When $m = 1$, (3.3) implies

$$\left| \int f\, d\mu - f(x_0) \right| \leqslant \begin{cases} w \left(\dfrac{d_r(x_0)}{h} \right), & r \geqslant 1, \\ w(c(x_0))^{1-r} \dfrac{d_r^r(x_0)}{h}, & r \leqslant 1. \end{cases} \tag{4.1}$$

If $w = \omega_1(f, h)$ the modulus of continuity of f in $[a, b]$, and $r \geqslant 1$, (4.1) becomes

$$\left| \int f\, d\mu - f(x_0) \right| \leqslant \omega_1(f, h) \frac{d_r(x_0)}{h}, \tag{4.2}$$

which in case $d_r(x_0) = l \cdot h$, $l \geqslant 1$, turns out to be

$$\left| \int f\, d\mu - f(x_0) \right| \leqslant l \cdot \omega_1 \left(f, \frac{1}{l} \cdot d_r(x_0) \right). \tag{4.3}$$

Note that inequality (4.2) is sharp when $r = 1$, namely, equality is attained by $f(t) = |t - x_0|$ where both of sides are $d_1(x_0)$.

COROLLARY 5. *For $m = 1$ and $h = d_2(x_0) \leqslant \min(x_0 - a, b - x_0)$* we have

$$\left| \int f\, d\mu - f(x_0) \right| \leqslant \omega_1(f, d_2(x_0)). \tag{5.1}$$

This is also true for $f \in C_B(\mathbb{R})$ (the space of real, bounded, continuous functions on $(-\infty, \infty)$) when $h = d_2(x_0) < \infty$.

Proof. Obvious from (4.2). ∎

Proof of Theorem 3. Let $g(t)=f(t)-f(x_0)$. From Lemma 2 we have

$$|g(t)|\leqslant\frac{w}{h}|t-x_0|.$$

Thus

$$\left|\int f\,d\mu-f(x_0)\right|=\left|\int g\,d\mu+(m-1)f(x_0)\right|\leqslant\int|g|\,d\mu+|m-1|\cdot|f(x_0)|,$$

i.e.,

$$\left|\int f\,d\mu-f(x_0)\right|\leqslant|m-1|\cdot|f(x_0)|+\frac{w}{h}\int|t-x_0|\,\mu(dt). \tag{3.4}$$

Here, equality holds for $f(t)=(w/h)\,|t-x_0|$ which fulfills the assumptions of the theorem.

The best constant θ in (3.4) is given by

$$\theta=\sup_{\mu}\int|t-x_0|\,\mu(dt),$$

where μ ranges over all measures on $[a,b]$ of mass m satisfying (3.1).

Letting $\gamma=m^{-1}\mu$ we determine

$$U=\sup_{\gamma}\int|t-x_0|\,\gamma(dt),$$

where γ ranges over all probability measures on $[a,b]$ satisfying

$$\int|t-x_0|^r\cdot\gamma(dt)=d_r^r(x_0)/m.$$

Note that $0\leqslant|t-x_0|\leqslant c(x_0)=\max(x_0-a,b-x_0)$. Taking the probability measure ρ induced by γ and the mapping $t\to|t-x_0|$ and denoting $u=|t-x_0|$, we seek

$$U=\sup_{\rho}\int u\,\rho(du)\qquad(0\leqslant u\leqslant c(x_0))$$

over all probability measures ρ such that

$$\int u^r\cdot\rho(du)=d_r^r(x_0)/m.$$

It follows (see [10, 11]) that

$$U=\phi(d_r^r(x_0)/m),$$

where

$$\Gamma_1=\{(z,\phi(z)) : 0\leqslant z\leqslant c^r(x_0)\}$$

is the upper boundary of the convex hull of the curve

$$\Gamma_0=\{(u^r,u) : 0\leqslant u\leqslant c(x_0)\}.$$

When $r\geqslant 1$, Γ_0 is concave and

$$U=d_r(x_0)/m^{1/r},$$

while, when $r<1$, Γ_0 is convex and

$$U=\frac{d_r^r(x_0)}{m}\,(c(x_0))^{1-r}.$$

As a result we get the best upper bound

$$\left|\int f\,d\mu-f(x_0)\right|\leqslant |m-1|\cdot|f(x_0)|+\frac{w}{h}\,\theta,$$

which completes the proof of the theorem. ∎

An application of Corollary 5 is

COROLLARY 6. *Let $f\in C_B[0,\infty)$ be such that $|f(t)-f(x_0)|$ is a convex function of t for a fixed $x_0\geqslant 1$. Consider the Szász–Mirakjan operator applied to f at x_0:*

$$(U_nf)(x_0)=e^{-n\cdot x_0}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{(n\cdot x_0)^k}{k!}.$$

Then

$$|(U_nf)(x_0)-f(x_0)|\leqslant \omega_1\left(f,\left(\frac{x_0}{n}\right)^{1/2}\right).$$

Proof. Consider $(X_j)_{j\in\mathbb{N}}$, Poisson (i.i.d.) random variables with parameter $x_0\geqslant 1$, so that $E(X)=\mathrm{Var}(X)=x_0$. Put $S_n=\sum_{j=1}^n X_j$, $n\geqslant 1$; then $E(S_n/n)=x_0$ and $\mathrm{Var}(S_n/n)=x_0/n$. Note that $\sqrt{x_0/n}\leqslant x_0$, so we can apply inequality (5.1) for $\mu=F_{S_n/n}$, the distribution function of S_n/n. ∎

For differentiable functions we have

THEOREM 7. *Let* $r > 0$, μ *a finite measure on* $[a, b] \subset \mathbb{R}$, $x_0 \in (a, b)$ *and* $c(x_0) = \max(x_0 - a, b - x_0)$. *Put*

$$\begin{aligned} c_k(x_0) &= \int (t - x_0)^k \, \mu(dt), \qquad k = 0, 1, \ldots, n; \\ d_r(x_0) &= \left(\int |t - x_0|^r \cdot \mu(dt) \right)^{1/r}. \end{aligned} \tag{7.1}$$

Let $f \in C^n[a, b]$, $n \geq 1$, *and assume* $|f^{(n)}(t) - f^{(n)}(x_0)|$ *is convex in* t *and*

$$|f^{(n)}(s) - f^{(n)}(t)| \leq w \qquad \text{if } s, t \in [a, b] \text{ and } |s - t| \leq h. \tag{7.2}$$

Here $0 < h \leq \min(x_0 - a, b - x_0)$ *and* $w > 0$ *are fixed.*
Then

$$\begin{aligned} E(x_0) &= \left| \int f \, d\mu - f(x_0) \right| - |f(x_0)| \cdot |c_0(x_0) - 1| - \sum_{k=1}^{n} \frac{|f^{(k)}(x_0)|}{k!} \cdot |c_k(x_0)| \\ &\leq \begin{cases} \dfrac{w}{h(n+1)!} \, d_r^{n+1}(x_0) \, c_0(x_0)^{1-((n+1)/r)}, & r \geq n+1 \\ \dfrac{w}{h(n+1)!} \, d_r^r(x_0)(c(x_0))^{(n+1)-r}, & r \leq n+1. \end{cases} \end{aligned} \tag{7.3}$$

Note. When $r = n+1$ and $w = \omega_1(f^{(n)}, h)$,

$$E(x_0) \leq \frac{\omega_1(f^{(n)}, h)}{h(n+1)!} \, d_{n+1}^{n+1}(x_0), \tag{7.4}$$

which, for $h = d_{n+1}^{n+1}(x_0)/(n+1)!$, becomes

$$E(x_0) \leq \omega_1 \left(f^{(n)}, \frac{d_{n+1}^{n+1}(x_0)}{(n+1)!} \right). \tag{7.5}$$

Inequality (7.4) is sharp; equality is attained by the function

$$\tilde{f}(t) = \begin{cases} \dfrac{(t - x_0)^{n+1}}{(n+1)!}, & x_0 \leq t \leq b, \\ 0, & a \leq t \leq x_0, \end{cases}$$

when $b-x_0 \geqslant x_0-a$, and by the function

$$\tilde{\tilde{f}}(t)=\begin{cases} \dfrac{(x_0-t)^{n+1}}{(n+1)!}, & a \leqslant t \leqslant x_0, \\ 0, & x_0 \leqslant t \leqslant b, \end{cases}$$

when $b-x_0 \leqslant x_0-a$.

In the first case an optimal measure μ_{x_0} is of mass $c_0(x_0)$, supported by $\{x_0, b\}$ and in the second case it is of the same mass $c_0(x_0)$, supported by $\{x_0, a\}$. In both cases the corresponding masses are $[c_0(x_0)-(d_{n+1}(x_0)/c(x_0))^{n+1}]$ and $(d_{n+1}(x_0)/c(x_0))^{n+1}$.

Remark 8. When $r=n$ and $w=\omega_1(f^{(n)}, h)$, inequality (7.3) becomes

$$\begin{aligned} E(x_0) &\leqslant \frac{\omega_1(f^{(n)}, h)}{h(n+1)!}\, d_n^n(x_0)\, c(x_0) \\ &= \frac{\omega_1(f^{(n)}, h)}{h}\left(\frac{d_n(x_0)}{c(x_0)}\right)^n \frac{(c(x_0))^{n+1}}{(n+1)!}. \end{aligned} \tag{8.1}$$

This is also sharp and equality is attained as in (7.4).

Proof of Theorem 7.

$$f(t)=\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(t-x_0)^k + I_t, \tag{7.6}$$

where

$$I_t=\int_{x_0}^{t}\left(\int_{x_0}^{t_1}\cdots\left(\int_{x_0}^{t_{n-1}}(f^{(n)}(t_n)-f^{(n)}(x_0))\,dt_n\right)\cdots\right)dt_1.$$

By Lemma 2,

$$|f^{(n)}(t)-f^{(n)}(x_0)| \leqslant \frac{w}{h}\cdot|t-x_0|,$$

$$|I_t| \leqslant \frac{w}{h}\cdot\frac{|t-x_0|^{n+1}}{(n+1)!}.$$

From (7.6), integrating relative to μ, we get

$$\begin{aligned} \left|\int f\,d\mu - f(x_0)\right| &\leqslant |f(x_0)|\cdot|c_0(x_0)-1| + \sum_{k=1}^{n}\frac{|f^{(k)}(x_0)|}{k!}\cdot|c_k(x_0)| \\ &\quad + \frac{w}{h(n+1)!}\int |t-x_0|^{n+1}\,\mu(dt). \end{aligned}$$

We would like to find

$$\theta = \sup_{\mu} \int |t - x_0|^{n+1} \mu(dt)$$

over all measures μ on $[a, b]$ of mass $c_0(x_0)$ with $(\int |t - x_0|^r \mu(dt))^{1/r} = d_r(x_0)$, when $d_r(x_0) > 0$.

Equivalently, we want

$$U = \sup_{\gamma} \int |t - x_0|^{n+1} \gamma(dt) \qquad (\theta = c_0(x_0)\, U)$$

over all probability measures $\gamma = m^{-1} \mu$ such that

$$\int |t - x_0|^r \gamma(dt) = d_r^r(x_0)/c_0(x_0).$$

Note that $0 \leqslant |t - x_0| \leqslant c(x_0) = \max(x_0 - a,\ b - x_0)$. Let ρ be the probability measure induced by γ and the mapping $t \to |t - x_0|$ and let $u = |t - x_0|$; we want to find

$$U = \sup_{\rho} \int u^{n+1} \rho(du) \qquad (0 \leqslant u \leqslant c(x_0)),$$

where ρ runs over all probability measures on $[0, c(x_0)]$ such that

$$\int u^r \rho(du) = d_r^r(x_0)/c_0(x_0).$$

From [10, 11] it follows that

$$U = \psi(d_r^r(x_0)/c_0(x_0)),$$

where $\{(z, \psi(z)) \colon 0 \leqslant z \leqslant c^r(x_0)\}$ is the upper boundary of the convex hull of the curve

$$G_0 = \{(u^r, u^{n+1}) \colon 0 \leqslant u \leqslant c(x_0)\}.$$

When $r \geqslant n+1$, G_0 is concave and

$$U = d_r^{n+1}(x_0)/(c_0(x_0))^{(n+1)/r},$$

while, when $r < n+1$, G_0 is convex and

$$U = \frac{d_r^r(x_0)}{c_0(x_0)} (c(x_0))^{(n+1-r)}.$$

Note that, for $r=n+1$, we find

$$U=d_{n+1}^{n+1}(x_0)/c_0(x_0).$$

Thus we get the upper bound

$$\left|\int f\,d\mu-f(x_0)\right|\leqslant |f(x_0)|\cdot|c_0(x_0)-1|+\sum_{k=1}^{n}\frac{|f^{(k)}(x_0)|}{k!}\cdot|c_k(x_0)|$$
$$+\frac{w}{h(n+1)!}\,\theta.$$

This completes the proof of the theorem. ∎

COROLLARY 9. *Let* $x_0\in(a,b)$ *and* $f\in C^1([a,b])$ *be such that* $|f'(t)-f'(x_0)|$ *is a convex function of* t. *Let* μ *be a probability measure on* $[a,b]$ *for which* $\int t\,\mu(dt)=x_0$ *and*

$$\left(\int (t-x_0)^2\,\mu(dt)\right)^{1/2}=d_2(x_0)>0.$$

If $d_2^2(x_0)\leqslant 2\min(x_0-a,b-x_0)$, *we get the sharp (attained) inequality*

$$\left|\int f\,d\mu-f(x_0)\right|\leqslant \omega_1(f',\tfrac{1}{2}\,d_2^2(x_0)). \tag{9.1}$$

And if $d_2(x_0)\leqslant 2\min(x_0-a,\,b-x_0)$, *we obtain the sharp inequality*:

$$\left|\int f\,d\mu-f(x_0)\right|\leqslant \omega_1(f',\tfrac{1}{2}\,d_2(x_0))\,d_2(x_0). \tag{9.2}$$

COROLLARY 10. *Let the random variable* X *have distribution* μ, $E(X)=x_0$ *and* $\mathrm{Var}(X)=\sigma^2<\infty$. *Consider those* $f\in C^1(\mathbb{R})$ *for which* $Ef(X)<\infty$ *and* $|f'(t)-f'(x_0)|$ *is convex in* t. *Then we have the sharp inequality (attained by* $f(t)=(t-x_0)^2$*)*:

$$|Ef(X)-f(x_0)|\leqslant \min\left\{\omega_1\left(f',\frac{\sigma^2}{2}\right),\,\omega_1\left(f',\frac{\sigma}{2}\right)\sigma\right\}. \tag{10.1}$$

The next result will be used in Theorem 12.

LEMMA 11. *Let* $x_0\in(a,b)\subset\mathbb{R}$, *and let* $c_1(x_0)$ *and* $c_0(x_0)>0$, $d_1(x_0)>0$ *be given numbers. Consider all measures* μ *on* $[a,b]$ *of mass* $c_0(x_0)$ *such that* $\int (t-x_0)\,\mu(dt)=c_1(x_0)$, $\int |t-x_0|\,\mu(dt)=d_1(x_0)$. *Put*

$$U(x_0)=\sup_{\mu}\int\frac{(t-x_0)^2}{c_0(x_0)}\,\mu(dt).$$

Then

$$U(x_0)=(b-x_0)\left(\frac{d_1(x_0)+c_1(x_0)}{2c_0(x_0)}\right)+(x_0-a)\left(\frac{d_1(x_0)-c_1(x_0)}{2c_0(x_0)}\right). \tag{11.1}$$

An optimal measure is supported by $\{a, x_0, b\}$.

Proof. Easy. ∎

Inequality (7.3) can be improved if we know more about μ. One result in this direction is

THEOREM 12. *Under the hypothesis of Lemma* 11, *let* $f\in C^1([a,b])$ *with* $\omega_1(f',h)\leqslant w$, *where* w, h *are given positive numbers such that* $0<h\leqslant\min(x_0-a, b-x_0)$. *Suppose* $|f'(t)-f'(x_0)|$ *is a convex function of* t. *Then*

$$\left|\int f\,d\mu-f(x_0)\right|\leqslant|f(x_0)|\cdot|c_0(x_0)-1|+|f'(x_0)|\cdot|c_1(x_0)| + \frac{w}{2h}c_0(x_0)\,U(x_0). \tag{12.1}$$

This inequality gives a best upper bound.

Inequality (12.1) is sharp, namely, it is attained when $c_0(x_0)=1$, $c_1(x_0)=0$, $\omega_1(f', h)=w$, $f(t)=(t-x_0)^2/2$ and μ is the probability measure supported by $\{a, x_0, b\}$ with masses

$$\frac{d_1(x_0)}{2(x_0-a)}, \left[1-\frac{d_1(x_0)}{2(x_0-a)}-\frac{d_1(x_0)}{2(b-x_0)}\right], \frac{d_1(x_0)}{2(b-x_0)},$$

respectively.

Proof. Easy. ∎

Note. When $n=r=1$, inequality (12.1) is better than the corresponding inequality (7.3).

An application to Corollary 9 is

COROLLARY 13. *Let* $f\in C^1([0, 1])$ *be such that* $|f'(t)-f'(x_0)|$ *is a convex function of* t, *let* $x_0\in(0, 1)$ *and consider the nth Bernstein operator applied to* f *at* x_0: $(B_nf)(x_0)=\sum_{k=0}^{n}f(k/n)\binom{n}{k}x_0^k(1-x_0)^{n-k}$. *Then*

(i) $$|(B_nf)(x_0)-f(x_0)| \leqslant\omega_1\left(f', \frac{x_0(1-x_0)}{2n}\right)\leqslant\omega_1\left(f', \frac{1}{8n}\right), \tag{13.1}$$

(ii)
$$\begin{aligned}|(B_nf)(x_0)-f(x_0)| &\leqslant \omega_1\left(f', \frac{1}{2}\sqrt{\frac{x_0(1-x_0)}{n}}\right)\sqrt{\frac{x_0(1-x_0)}{n}}\\ &\leqslant \omega_1\left(f', \frac{1}{4\sqrt{n}}\right)\frac{1}{2\sqrt{n}}, \qquad x_0 \in \left[\frac{1}{5}, \frac{4}{5}\right].\end{aligned} \tag{13.2}$$

Proof. (i) Let $(X_j)_{j\in\mathbb{N}}$ be Bernoulli (i.i.d.) random variables such that $P(X_j=0)=1-x_0$, $P(X_j=1)=x_0$; then $E(X)=x_0$, $\mathrm{Var}(X)=x_0(1-x_0)$. Put $S_n=\sum_{j=1}^n X_j$, $n\geqslant 1$; then $E(S_n/n)=x_0$ and $\mathrm{Var}(S_n/n)=x_0(1-x_0)/n<2\min(x_0, 1-x_0)$. Now apply inequality (9.1) to $\mu=F_{S_n/n}$, the distribution function of S_n/n. Further, note that $\max_{0<x_0<1}(x_0(1-x_0))=\frac{1}{4}$, being attained at $x_0=\frac{1}{2}$.

(ii) Proved similarly. ∎

An application to Corollary 10 is

COROLLARY 14. *Let f be a real function, bounded and having a continuous bounded derivative on $(-\infty, \infty)$ and let $|f'(t)-f'(x_0)|$ be a convex function of t for some fixed $x_0\in\mathbb{R}$. Consider the nth Weierstrass operator applied to f at x_0:*

$$(W_nf)(x_0)=\sqrt{n/\pi}\int_{-\infty}^{\infty} f(x)\,e^{-n(x-x_0)^2}\,dx.$$

Then

$$|(W_nf)(x_0)-f(x_0)|\leqslant \min\left\{\omega_1\left(f', \frac{1}{4n}\right), \omega_1\left(f', \frac{1}{2\sqrt{2n}}\right)\frac{1}{\sqrt{2n}}\right\}. \tag{14.1}$$

Proof. As that of Corollary 13. Here the random variable X has the normal distribution $(x_0, \frac{1}{2})$ with density $(1/\sqrt{\pi})\,e^{-(x-x_0)^2}$. Then apply inequality (10.1). ∎

Remark 15. In Theorems 3, 7 and related results, when $x_0=0$ and $h\leqslant\min(|a|, b)$, we can use instead of ω_1

$$\begin{aligned}\bar{\omega}_1(f^{(m)},h)&=\sup\{|f^{(m)}(x)-f^{(m)}(y)|: x\cdot y\geqslant 0 \quad\text{and}\quad |x-y|\leqslant h\}\\ &<\omega_1(f^{(m)}, h), \qquad \text{for and integer } m\geqslant 0.\end{aligned}$$

4. Multidimensional Results

Definition 16. For f a continuous real valued function on a compact subset Q of $\mathbb{R}^k$, $k \geqslant 1$, its modulus of continuity is

$$\omega_1(f, h) = \sup\{|f(\mathbf{x}) - f(\mathbf{y})|: \text{all } \mathbf{x},\ \mathbf{y} \in Q,\ \|\mathbf{x} - \mathbf{y}\| \leqslant h\},$$

where $\|\cdot\|$ is a norm in $\mathbb{R}^k$.

Theorem 17. *Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \geqslant 1$, let $\mathbf{x}_0 = (x_{01}, ..., x_{0k}) \in Q$ be fixed and let μ be a probability measure on Q. Let $f \in C^n(Q)$, $n \geqslant 1$, and suppose that each nth partial derivative $f_\alpha = \partial^\alpha f / \partial x^\alpha$, where $\alpha = (\alpha_1, ..., \alpha_k)$, $\alpha_i \geqslant 0$, $i = 1, ..., k$, and $|\alpha| = \sum_{i=1}^k \alpha_i = n$ has, relative to Q and the l_1-norm, a modulus of continuity $\omega_1(f_\alpha, h) \leqslant w$, and each $|f_\alpha(\mathbf{x}) - f_\alpha(\mathbf{x}_0)|$ is a convex function of $\mathbf{x}$. Here h and w are given positive numbers, and h is chosen so that the ball in $\mathbb{R}^k$: $B(\mathbf{x}_0, h)$ is contained in Q. Then*

$$\left|\int_Q f\, d\mu - f(\mathbf{x}_0)\right| \leqslant \left|\sum_{j=1}^n \frac{1}{j!} \int_Q g_{\mathbf{x}}^{(j)}(0)\, \mu(d\mathbf{x})\right| + \frac{w}{h(n+1)!} \int_Q \|x - x_0\|^{n+1}\, \mu(d\mathbf{x}), \tag{17.1}$$

where $g_{\mathbf{x}}(t) = f(\mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0))$, $t \geqslant 0$.

Proof.

$$f(z_1, ..., z_k) = g_{\mathbf{z}}(1) = \sum_{j=0}^n \frac{g_{\mathbf{z}(0)}^{(j)}}{j!} + R_n(\mathbf{z}, 0), \tag{17.2}$$

where

$$g_{\mathbf{z}}^{(j)}(t) = \left[\left(\sum_{i=1}^k (z_i - x_{0i}) \frac{\partial}{\partial x_i}\right)^j f\right](x_{01} + t(z_1 - x_{01}), ..., x_{0k} + t(z_k - x_{0k})) \tag{17.3}$$

is the jth derivative of $g_{\mathbf{z}}(t) = f(\mathbf{x}_0 + t(\mathbf{z} - \mathbf{x}_0))$ and

$$R_n(\mathbf{z}, 0) = \int_0^1 \left(\int_0^{t_1} \cdots \left(\int_0^{t_{n-1}} (g_{\mathbf{z}}^{(n)}(t_n) - g_{\mathbf{z}}^{(n)}(0))\, dt_n\right) \cdots\right) dt_1.$$

By Lemma 2 we get

$$|f_\alpha(\mathbf{x}_0 + t(\mathbf{z} - \mathbf{x}_0)) - f_\alpha(\mathbf{x}_0)| \leqslant \frac{w}{h} t\, \|\mathbf{z} - \mathbf{x}_0\|, \qquad \text{all} \quad t \geqslant 0.$$

It follows from (17.3) that

$$|R_n(\mathbf{x}, 0)|$$
$$\leqslant \int_0^1 \left[\int_0^{t_1} \cdots \left[\int_0^{t_{n-1}} \left(\sum_{|\alpha|=n} \frac{n! \prod_{i=1}^k |z_i - x_{0i}|^{\alpha_i}}{\alpha_1! \cdots \alpha_k!} \frac{w}{h} \|\mathbf{z}-\mathbf{x}_0\| \, t_n \right) \cdot dt_n \right] \cdots \right] \cdot dt_1$$
$$= \frac{w}{h} \frac{\|\mathbf{z}-\mathbf{x}_0\|^{n+1}}{(n+1)!}.$$

Therefore

$$|R_n(\mathbf{z}, 0)| \leqslant \frac{w}{h} \frac{\|\mathbf{z}-\mathbf{x}_0\|^{n+1}}{(n+1)!}, \qquad \text{for all } \mathbf{z} \in Q. \tag{17.4}$$

Note $g_{\mathbf{z}}(0) = f(\mathbf{x}_0)$. Integrating (17.2) relative to μ and using (17.4), (17.1) follows. ∎

Remark 18. Let n be even and let Q be the closed line segment in $\mathbb{R}^k$ $(k \geqslant 1)$ joining

$$(\underbrace{-1, -1, ..., -1}_{k}) \qquad \text{to} \qquad (\underbrace{1, 1, ..., 1}_{k}).$$

Let $\mathbf{x}_0 = \mathbf{0}$, let $0 < h \leqslant 1$ and take $w = \max\{\omega_1(f_\alpha, h)$: all α such that $|\alpha| = n\}$. For $f(\mathbf{x}) = \|\mathbf{x}\|^{n+1}/(n+1)!$, $\|\cdot\|$ the l_1 norm in $\mathbb{R}^k$, equality is attained in (17.1).

Namely, all $f_\alpha(\mathbf{x}) = \|\mathbf{x}\|$ are convex functions so that $\omega_1(f_\alpha, h) = h$ and $g_{\mathbf{z}}^{(j)}(0) = 0$ for all $0 \leqslant j \leqslant n$.

Illustration 19. (i) For $Q = \{\mathbf{x} \in \mathbb{R}^k \colon \|\mathbf{x}\|_{l_1} \leqslant 1\}$ we have

$$\left| \int_Q f \, d\mu - f(\mathbf{x}_0) \right| \leqslant \left| \sum_{j=1}^n \frac{1}{j!} \int_Q g_{\mathbf{x}}^{(j)}(0) \, \mu(d\mathbf{x}) \right| + \frac{w}{h} \frac{(1 + \|\mathbf{x}_0\|)^{n+1}}{(n+1)!}, \tag{19.1}$$

where h is such that $B(\mathbf{x}_0, h) \subset Q$.

(ii) For $Q = \{\mathbf{x} \in \mathbb{R}^k \colon -\lambda \leqslant x_i \leqslant \lambda,\ i = 1, ..., k\}$, $\lambda > 0$, we get

$$\left| \int_Q f \, d\mu - f(\mathbf{x}_0) \right| \leqslant \left| \sum_{j=1}^n \frac{1}{j!} \int_Q g_{\mathbf{x}}^{(j)}(0) \, \mu(d\mathbf{x}) \right| + \frac{w}{h} \frac{(\|\mathbf{x}_0\| + k\lambda)^{n+1}}{(n+1)!}, \tag{19.2}$$

where $\|\cdot\|$ is the l_1 norm in $\mathbb{R}^k$ and $B(\mathbf{x}_0, h) \subset Q$.

COROLLARY 20. *Let the random vector* $(X_1, ..., X_k)$ *take values in a convex subset* Q *of* $\mathbb{R}^k$, *with distribution function* μ *and expectations* $E(X_i) = x_{0i}$, $i = 1, ..., k$. *Thus* $\mathbf{x}_0 = (x_{01}, ..., x_{0k})$ *is a fixed point of* Q. *Further, put* $(\int \|\mathbf{x} - \mathbf{x}_0\|^2 \mu(d\mathbf{x}))^{1/2} = \sigma$, *where.* $\|\cdot\|$ *denotes the* l_1 *norm in* $\mathbb{R}^k$. *Let* f *have continuous first order partial derivatives on* Q, *let* f *and these derivatives be bounded on* Q *and let* $|f_i(\mathbf{x}) - f_i(\mathbf{x}_0)|$ *be a convex function of* $\mathbf{x}$ *for* $i = 1, ..., k$, *where* f_i *is the ith first partial derivative of* f. *For* $h > 0$ *set*

$$\omega_1^*(f_i, h) = \max\{\omega_1(f_i, h) : i = 1, ..., k\}.$$

If $h = \sigma^2/2$ *and* $B(\mathbf{x}_0, \sigma^2/2) \subset Q$, *then*

$$\left| \int_Q f \, d\mu - f(\mathbf{x}_0) \right| \leqslant \omega_1^* \left(f_i, \frac{\sigma^2}{2} \right). \tag{20.1}$$

If $h = \sigma/2$ *and* $B(\mathbf{x}_0, \sigma/2) \subset Q$, *then*

$$\left| \int_Q f \, d\mu - f(\mathbf{x}_0) \right| \leqslant \omega_1^* \left(f_i, \frac{\sigma}{2} \right) \sigma. \tag{20.2}$$

Proof. Apply Theorem 17 with $n = 1$ and $w = \omega_1^*(f_i, h)$. ∎

Remark 21. Let r, $C(\mathbf{x}_0)$, $D_r(\mathbf{x}_0)$ be given positive numbers. Assume the convex and compact set Q lies in the ball $0 \leqslant \|\mathbf{x} - \mathbf{x}_0\|_{l_1} \leqslant C(\mathbf{x}_0)$ and that the probability measure μ on Q satisfies $(\int \|\mathbf{x} - \mathbf{x}_0\|_{l_1}^r \cdot \mu(d\mathbf{x}))^{1/r} = D_r(\mathbf{x}_0)$. Then, using standard moment methods, we find that the remainder term on the right-hand side of (17.1) is $\leqslant (w/h(n+1)!) \, D_r^{n+1}(\mathbf{x}_0)$ if $r \geqslant n+1$, and $\leqslant (w/h(n+1)!) \; D_r^r(\mathbf{x}_0)(C(\mathbf{x}_0))^{(n+1)-r}$ if $r \leqslant n+1$. Thus we have generalized (7.3) to higher dimension.

As a further result we have

PROPOSITION 22. *Take* Q *a convex and compact subset of* $\mathbb{R}^k$, *let* $\mathbf{x}_0 = (x_{01}, ..., x_{0k}) \in Q$ *be fixed and let* μ *be a probability measure on* Q. *Let* $f \in C(Q)$, $|f(\mathbf{x}) - f(\mathbf{x}_0)|$ *being a convex function of* $\mathbf{x}$. *Assume* f *has, relative to* Q *and a norm* $\|\cdot\|$ *in* $\mathbb{R}^k$, *a modulus of continuity for which* $\omega_1(f, h) \leqslant w$. *Here* h, w *are given positive numbers such that the ball* $B(\mathbf{x}_0, h) \subset Q$. *Then*

$$\left| \int_Q f \, d\mu - f(\mathbf{x}_0) \right| \leqslant \frac{w}{h} \left(\int_Q \|\mathbf{x} - \mathbf{x}_0\| \, \mu(d\mathbf{x}) \right). \tag{22.1}$$

This inequality is sharp; equality is attained by $f(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_0\|$ *when* $w = \omega_1(f, h)$.

Proof. Obvious, using Lemma 2. ∎

Remark 23. Assume the first two sentences of Remark 21, except that instead of $\|\cdot\|_{l_1}$ take any norm in $\mathbb{R}^k$. Then, using standard moment methods, we obtain *a* best upper bound:

$$\left|\int_Q f\,d\mu - f(\mathbf{x}_0)\right| \leqslant \begin{cases} \frac{w}{h} D_r(\mathbf{x}_0), & \text{if } r \geqslant 1, \\ \frac{w}{h} D_r^r(\mathbf{x}_0)(C(\mathbf{x}_0))^{1-r}, & \text{if } r \leqslant 1. \end{cases} \tag{23.1}$$

This is a generalization of inequality (3.3).

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